# Exactly solvable models in non-equilibrium

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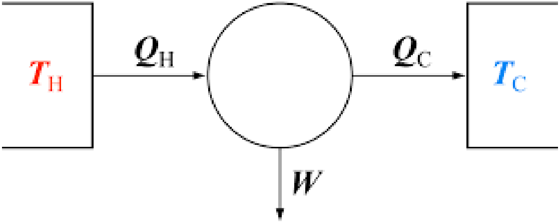
# Outline

- Models of interacting particle systems (IPS) are used to understand systems of non-equilibrium statistical physics.
- Some of these IPS have extra structure which allows to produce exact formulas for quantities such as density or temperature profile, correlation functions.
- This extra structure comes a special property called "duality" or "self-duality", which enables to connect the model of interest to another, simpler, dual one, via a duality function.

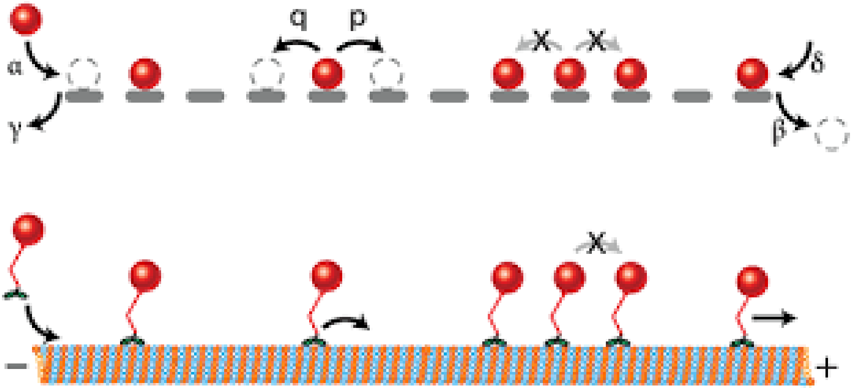
- A model and its dual turn out to be two different representations of an abstract element of a Lie-algebra.
- In this way, we can classify and constructively produce such systems, starting from the underlying Lie-algebras. IPS with duality properties come in families, associated to Lie-algebras.
- This constructive algebraic approach is robust enables to find all the duality functions, including orthogonal ones.

- For symmetric (=detailed balance in the bulk) systems, these algebras generating families of models with duality are classical Lie-algebras such as $SU(2), SU(n), SU(1,1)$, Heisenberg.
- The "correct" asymmetric companion model of the symmetric models, can be found via $q$-deformation of these algebras, i.e., the corresponding quantum Lie algebras, where $0 < q < 1$, models the asymmetry.
- Remark: Duality and self-duality is strictly weaker than integrability.

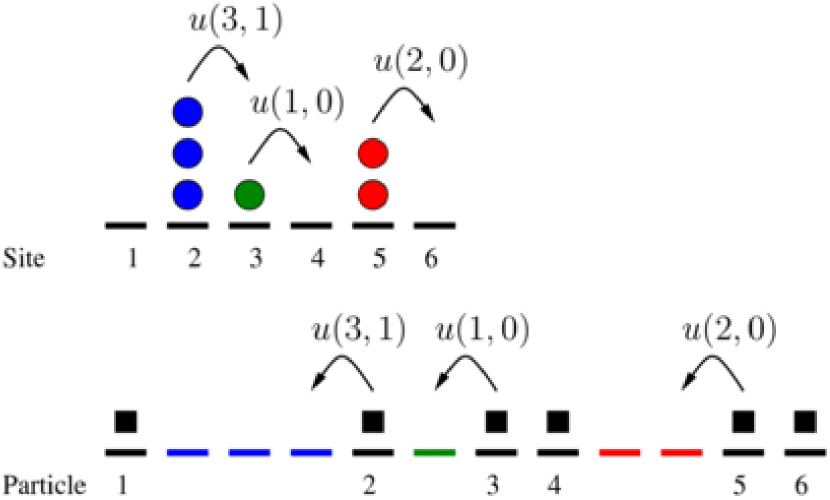
# Non-equilibrium



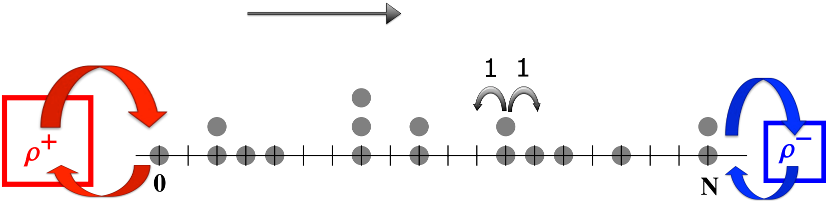
# IPS1



# IPS2

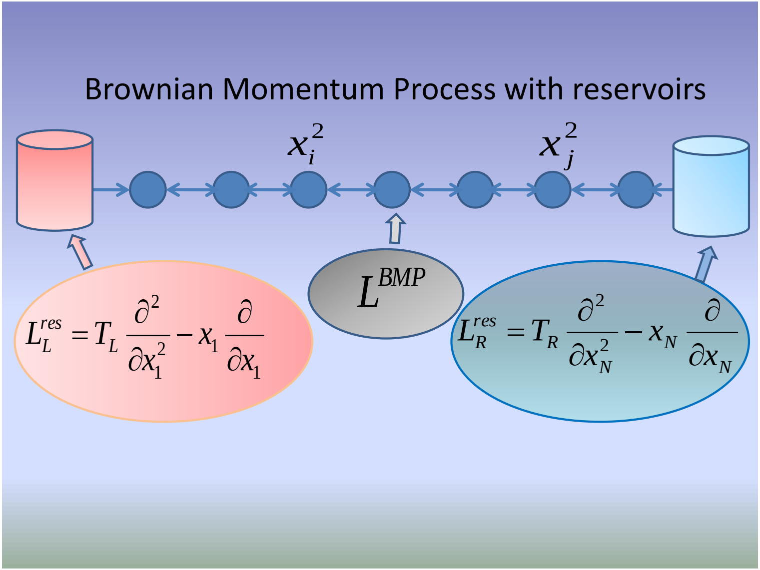


# IPS in modelling non-equilbrium

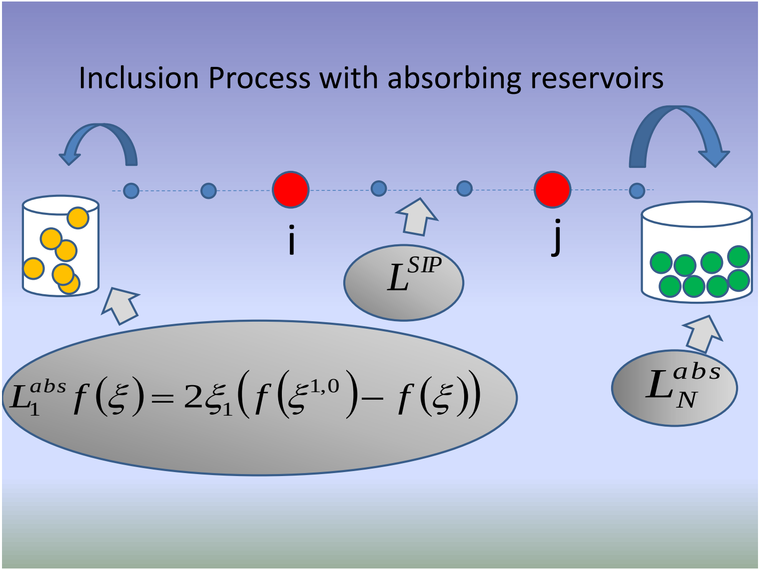


# Duality

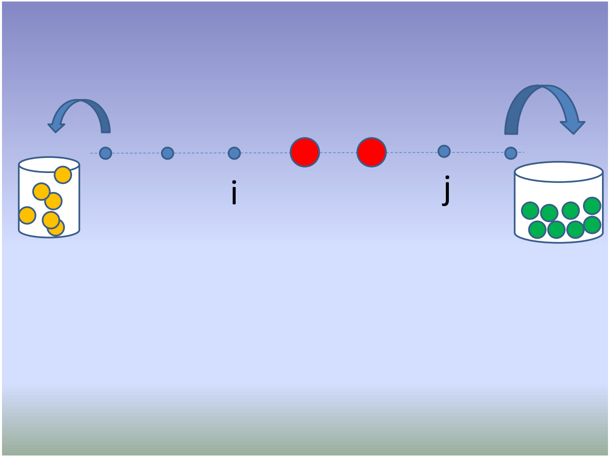
# Non equilibrium model



# Non equilibrium model: the dual



# Non equilibrium model: the computation of covariances reduces to two particle absorption probabilities in the dual



# Duality Families

# Heisenberg algebra family

# $SU(2)$ family

# $SU(1,1)$ family

# Non-equilibrium

- Boundary driven non-equilibrium: Equilibrium (reversible, detailed balance) systems coupled to reservoirs at em different temperatures, different chemical potentials.
- Bulk driven non-equilibrium: Systems driven by a bulk field, pushing particles in a preferred direction.
- Active particles: particles with intrinsic degree of freedom coupled to direction of motion.
- Various combinations of the previous.

Aims: understanding nature of the stationary state (NESS), density profile, (long-range) correlations, currents, analogues of free energy (large deviations), non-equilibrium fluctuation symmetries.

# Interacting particle systems (IPS)

- Stochastic, Markovian dynamics aiming at modelling complex systems of interacting units defined on a discrete vertex set.
- Important subfield of probability theory, initiated by Spitzer (1970), Dobrushin (1968). First overview book Liggett (1985). From 1980 up to now one of leading subfields of probability theory. Connections with physics, biology, computer science.
- Well studied IPS: SEP (symmetric exclusion process), ASEP (asymmetric exclusion process), Voter model (opinion dynamics, interfaces), Contact process (spread of infection), KMP model (model of heat conduction).

- IPS in statistical physics: meta-stability, phase transitions, non-equilibrium steady states, micro-to-macro transitions (hydrodynamic limits).
- IPS in population biology: evolution of traits in a population under mutation, migration, selection, territory of competing species.
- IPS in mathematics: to give sense to a priori ill-defined stochastic partial differential equations such as the KPZ (Kardar, Parisi, Zhang) equation, Allen-Cahn equation etc.

# Background: Markov processes and their generators

A Markov process is a process for which the distribution of future states only depends on the current state, and not on the previous history. Important example: random walk (versus non-Markovian walks such as self-avoiding random walk, self-reinforced random walk), or in the continuum Brownian motion (versus other non-Markovian Gaussian processes such as fractional Brownian motion).

# Generator

Because of this Markov property the process is totally determined by a single operator, the so-called generator $L$.

$$\mathbb{E}(f(X_t)|X_0 = x) = f(x) + t(Lf)(x) + O(t^2)$$

The relation between expectation in the course of time in the Markov process and the generator is by "exponentiation"

$$\mathbb{E}_x f(X_t) := \mathbb{E}(f(X_t)|X_0 = x) = e^{tL} f(x)$$

where the exponential is defined via the Hille-Yoshida theorem.

# Generator of diffusion processes

The generator of diffusion processes (drift+noise) of the form

$$dX_t = b(X_t)dt + \sqrt{2a}dW_t$$

where $\{W_t, t \geq 0\}$ is Brownian motion reads

$$Lf = \sum_i b_i(x)\partial_i f + \sum_{ij} a_{ij}(x)\partial^2_{ij} f$$

i.e., second order differential operator with $a_{ij}$ positive definite matrix. For Brownian motion $L = \frac{1}{2}\Delta$.

# Generator of jump processes

For jump processes the generator is of the form

$$Lf(\eta) = \sum_{\eta'} c(\eta, \eta')(f(\eta') - f(\eta))$$

where $c(\eta, \eta')$ is the rate of jump from $\eta$ to $\eta'$.
E.g. rate one Poisson process

$$Lf(n) = f(n+1) - f(n)$$

# Duality for Markov processes

Duality is a way to connect (intertwine) two Markov processes (a process and its dual) via a function, called the duality function.

$$\{\xi_t, t \geq 0\} \longrightarrow^D \{\eta_t, t \geq 0\}$$

if for all $\eta, \xi; t \geq 0$

$$\mathbb{E}(D(\xi_t, \eta)|\xi_0 = \xi) = \mathbb{E}(D(\xi, \eta_t)|\eta_0 = \eta)$$

I.e., evolving $\{\xi_t, t \geq 0\}$ and while fixing $\eta$ has the same effect as evolving $\{\eta_t, t \geq 0\}$ while fixing $\xi$. If the processes are the same, then we call it self-duality.

# Duality of generators

- Duality is equivalent with duality on the level of generators, i.e., if $L$ is the generator of $\{\xi_t, t \geq 0\}$ and $\mathcal{L}$ is the generator of $\{\eta_t, t \geq 0\}$

$$L_{\text{left}} D(\xi, \eta) = \mathcal{L}_{\text{right}} D(\xi, \eta)$$

- In finite or countable state space setting this can be rewritten in matrix notation as

$$LD = D\mathcal{L}^T$$

Notation

$$L \longrightarrow^D \mathcal{L}$$

# Duality and Lie algebras: a simple example

Consider the Wright-Fisher diffusion on $[0,1]$:

$$dX_t = \sqrt{2X_t(1-X_t)}dW_t$$

with generator

$$Lf = x(1-x)f''$$

and consider the discrete process on $\mathbb{N}$ with generator

$$\mathcal{L}f(n) = n(n-1)(f(n-1) - f(n))$$

which is a pure death chain $\{N_t, t \geq 0\}$, called the Kingman's coalescent (block-counting process).

One sees

$$Lx^n = x(1-x)\frac{d^2}{dx^2}(x^n) = n(n-1)(x^{n-1} - x^n)$$

which one can write in the form

$$L_{\text{right}} D(n,x) = \mathcal{L}_{\text{left}} D(n,x) \tag{1}$$

I.e., the function $D(n,x) = x^n$ is a duality function between the processes $\{X_t, t \geq 0\}$ and $\{N_t, t \geq 0\}$. For the processes, this means

$$\mathbb{E}_x^{WF}(X_t^n) = \mathbb{E}_n^{KC}(x^{N_t})$$

i.e., all the time dependent moments in the Wright Fisher diffusion can be calculated using the Kingman's coalescent.

# Heisenberg algebra

Two representations

$$Af(x) = f'(x),\ A^\dagger f(x) = xf(x) \tag{2}$$
$$af(n) = nf(n-1), a^\dagger f(n) = f(n+1) \tag{3}$$
$$\tag{4}$$

Then we have

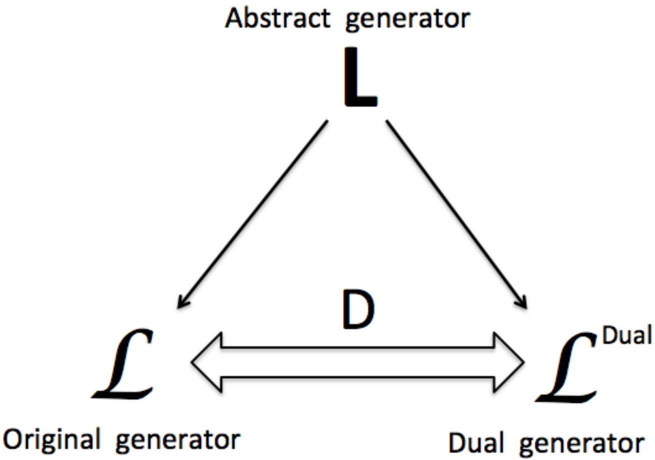
$$L_{WF} = A^\dagger (I - A^\dagger) A^2$$

$$\mathcal{L}_{KC} = a^2 (I - a^\dagger) a^\dagger$$

Moreover, the duality relation hold on the level of the $A, a, A^\dagger, a^\dagger$ operators:

$$A_{\text{right}} D(n,x) = a_{\text{left}} D(n,x); A^\dagger_{\text{right}} D(n,x) = a^\dagger_{\text{left}} D(n,x)$$

# Abstract generator



- Duality means that the generators are different representations of the same "abstract" object. This abstract object is an element of a Lie-algebra (in the example Heisenberg algebra).
- Duality then holds on the level of the algebra generators (in the example $a, a^\dagger, A, A^\dagger$. It corresponds to an intertwining between two representations, the duality function is the intertwiner)

# Duality, symmetries and detailed balance

- Duality + symmetry gives new duality. I.e., $L \longrightarrow^{D} \mathcal{L}$ and $[L, S] = LS - SL = 0$ ($S$ is a symmetry of $L$) then

$$L \longrightarrow^{D'} \mathcal{L}, \text{ with, } D' = S_{\text{left}} D$$

- Detailed balance gives a "cheap self-duality": if (in the case of Markov jump processes with rates $c(\eta, \eta')$

$$\mu(\eta) c(\eta, \eta') = \mu(\eta') c(\eta', \eta)$$

then

$$L \longrightarrow^{D} L, \text{ with } D(\xi, \eta) = \mu(\xi)^{-1} \delta_{\xi,\eta}$$

Self-duality is non-diagonal form of detailed balance.

# Generators of particle systems

- For the context of IPS we have state spaces of the form "configuration space": $\mathbb{R}^V$, $\mathbb{N}^V$, $\{0,1,\ldots,\alpha\}^V$. where $V$ is the vertex set: $V=\mathbb{Z}^d$, or a finite lattice $V_N=[-N,N]^d\cap\mathbb{Z}^d$ in case of systems coupled to boundaries. We focus on systems with a conserved quantity.
- The "bulk" generators are of "particle exchange along an edge type", i.e.,

$$L=\sum_{ij\in V} p(ij)(\mathcal{L}_{ij}+\mathcal{L}_{ji})$$

where $\mathcal{L}_{ij}+\mathcal{L}_{ji}$ is the single-edge generator associated to the edge $ij$. It describes e.g. hopping of particles over the edge, or transport of energy, momentum over the edge. The edge weight is symmetric $p(ij)=p(ji)$.

- This form is (not by chance) reminiscent of spin-chain Hamiltonians of the form

$$H = \sum_i H_{i,i+1}$$

such as fermionic or bosonic spin chains, e.g. $XXZ$ chain.

- In case of systems coupled to boundaries, e.g., $V = \{1, \ldots, N\}$ the generator is of the form

$$L = A_1 + A_N + \sum_{i=1}^{N} \mathcal{L}_{i,i+1}$$

where $A_1$, $A_N$ represent the left, resp. right boundary generators.

# Important examples of single edge generators (i).

- Symmetric Exclusion process (SEP): $SU(2)$ family. State space $\{0,1\}^V$.

$$\mathcal{L}_{ij} f(\eta) = \eta_i (1-\eta_j)(f(\eta - e_i + e_j) - f(\eta))$$

- Symmetric Inclusion process (SIP): $SU(1,1)$ family. State space $\mathbb{N}^{Zd}$

$$\mathcal{L}_{ij} f(\eta) = \eta_i (\alpha + \eta_j)(f(\eta - e_i + e_j) - f(\eta))$$

- Brownian energy process (BEP): $SU(1,1)$ family. State space $[0,\infty)^V$.

$$\mathcal{L}_{ij} f(x) = \left(-\alpha(x_i - x_j)(\partial_i - \partial_j) + x_i x_j (\partial_i - \partial_j)^2\right) f(x)$$

# Important examples of single edge generators (ii).

- Discrete KMP model: $SU(1,1)$ family. State space $\mathbb{N}^V$.

$$\mathcal{L}_{ij} f(\eta) = \sum_{k=0}^{\eta_i+\eta_j} \frac{1}{\eta_i + \eta_j + 1} (f(\eta + (k - \eta_i)e_i + (-\eta_i - \eta_j + k)e_j) - f(\eta))$$

- Continuous KMP model: $SU(1,1)$ family. State space $[0,\infty)^V$.

$$\mathcal{L}_{ij} f(\eta) = \mathbb{E}(f(\eta - \eta_i e_i - \eta_j e_j + U(\eta_i + \eta_j)e_i + (1-U)e_j) - f(\eta))$$

where $U$ is a uniform random variable on $[0,1]$.

# Important examples of single edge generators (iii).

- Independent random walks: Heisenberg family. State space $\mathbb{N}^V$.

$$\mathcal{L}_{ij} f(\eta) = \eta_i (f(\eta - e_i + e_j) - f(\eta))$$

- Averaging processes: Heisenberg family. State space $\mathbb{R}^V$

$$\mathcal{L}_{ij} f(\eta) = \eta_i \left( f(\eta - \eta_i e_i - \eta_j e_j + \frac{\eta_i + \eta_j}{2}(e_i + e_j)) - f(\eta) \right)$$

# What is duality good for in models of non-equilibrium?

It enables to simplify the study of the original process $\{\eta_t, t \geq 0\}$, via, e.g.

- Going from continuous to discrete variables (or vice-versa).
- Going from many to a few particles.
- Replacing boundary reservoirs by absorbing states.
- Proofs of existence of infinite particle systems (because the dual is finite).
- Going from an asymmetric system to a symmetric system.

- We are interested in finding systems for which the duality functions are in factorized form, i.e., of the form

$$D(\xi,\eta) = \prod_{i\in V} d(\xi_i,\eta_i)$$

the function $d$ is then called the single vertex duality function.
- The corresponding symmetries will then be in “exponential form”

$$S = \exp\left(\sum_{i\in V} S_i\right)$$

or equivalently, in additive form

$$S = \sum_{i\in V} S_i$$

# $SU(1,1)$ family

# Interacting particle systems from $SU(1,1)$: inclusion process, and related diffusion processes

- Inclusion process. state space $\mathbb{N}^{Zd}$

$$\mathcal{L}_{ij} f(\eta) = \eta_i(\alpha + \eta_j) f(\eta - e_i + e_j) - f(\eta)$$

- Brownian energy process. state space $[0, \infty)^V$.

$$\mathcal{L}_{ij} f = -\alpha(x_i - x_j)(\partial_i - \partial_j) + x_i x_j (\partial_i - \partial_j)^2$$

- Discrete KMP model. State space $\mathbb{N}^V$.

$$\mathcal{L}_{ij} f(\eta) = \sum_{k=0}^{\eta_i+\eta_j} \frac{1}{\eta_i + \eta_j + 1} (f(\eta + (k - \eta_i) e_i + (-\eta_i - \eta_j + k) e_j) - f(\eta))$$

- Abstract generator

$$L_{ij} = K_i^+ K_j^- + K_i^- K_j^+ - 2K_i^0 K_j^0 + \frac{\alpha^2}{2}$$

  with commutation relations

$$[K^\pm, K^0] = \pm K^\pm,\ [K^+, K^-] = +2K^0$$

- Symmetries: $K_i^a + K_j^a$, $a \in \{+, -, 0\}$.

# Asymmetric systems

- The systems we just considered are "bulk" symmetric, or detailed balance.
- To find the correct asymmetric versions of them is a challenge and works so far only in $d = 1$.
- The way to proceed is to consider the same abstract generators in the $q$-deformations of the corresponding algebras (i.e., considering the corresponding $q$-deformed quantum or Hopf algebras).
- Here for the deformation parameter $q = 1$ corresponds to the symmetric case $q = 0$ to the totally asymmetric case.

# Thanks for your attention