From constructive to tensor field theory

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1 Field Theory

2 On forests

3 Tensor field theories
Physics changes with the energy scale:
Renormalization Group

Flow in the space of theories ("time" $\sim$ energy scale) [Polchinski '84 ...]

Fixed points and trajectories
Weak versus strong coupling

Weak coupling
Perturbation theory

Strong coupling
Non perturbative?
1 Field Theory

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Jacques’s laws (around 2005)

First law

\[ f(1) = f(0) + \int_0^1 dt f'(t) \]

Too complicated!
<table>
<thead>
<tr>
<th>First law</th>
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<td>$0 = 0$</td>
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**First law**

\[ 0 = 0 \]

Too complicated!
First law

\[ 0 = 0 \]

Too complicated!

Second law

*The only formula you need to know is:*

\[ f(1) = f(0) + \int_0^1 dt f'(t) \]
For three variables...

$$f(x_{12}, x_{13}, x_{23})$$
For three variables...

\[ f(x_{12}, x_{13}, x_{23}) \]

apply (carefully!) the second law
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\[ f(x_{12}, x_{13}, x_{23}) \]

apply (carefully!) the second law

\[
f(1, 1, 1) = f(0, 0, 0) + \int_0^1 du_{12} \frac{\partial f}{\partial x_{12}}(u_{12}, 0, 0) + \ldots
\]

\[
+ \int_0^1 du_{12} du_{13} \frac{\partial^2 f}{\partial x_{12} \partial x_{13}}(u_{12}, u_{13}, \inf(u_{12}, u_{13})) + \ldots
\]
Consider the complete graph over $n$ vertices labelled \( \{1, \ldots, n\} \) and let \( f(x_{ij}) \) be a function of the \( \binom{n}{2} \) link variables \( x_{ij} \). Then

\[
f(1, \ldots, 1) = \sum_F \int_0^1 \left( \prod_{(k,l) \in F} du_{kl} \right) \left( \frac{\partial |F| f}{\prod_{(k,l) \in F} \partial x_{kl}} \right) (w_{ij}^F),
\]

- \( F \) runs over the forests (acyclic subgraphs) of the complete graph
- to each edge \((k, l)\) in the forest we associate a variable \( u_{kl} \) which is integrated from 0 to 1
- we take the derivative of \( f \) with respect to the variables associated to the edges in the forest
- we evaluate this derivative at \( x_{ij} = w_{ij}^F \), the infimum of \( u \) along the path in \( F \) connecting the vertices \( i \) and \( j \)
The $w^F_{ij}$ matrix

$$w^F = \begin{pmatrix}
1 & u_{12} & u_{13} & \inf(u_{13}, u_{34}) & 0 & 0 \\
\vdots & 1 & \inf(u_{12}, u_{13}) & \inf(u_{12}, u_{13}, u_{34}) & 0 & 0 \\
\vdots & \vdots & 1 & u_{34} & 0 & 0 \\
\vdots & \vdots & \vdots & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & 1 & u_{56} \\
\vdots & \vdots & \vdots & \vdots & \vdots & 1
\end{pmatrix} \geq 0!$$
Notes on the Brydges-Kennedy-Abdesselam-Rivasseau forest interpolation formula

There are many instances in mathematical physics where one tries to understand joint probability measures for a collection of random variables $X_1, \ldots, X_n$ with $n$ large, of the form

$$e^{-\sum_{i=1}^{n} V(x_i)} d\mu_C(x),$$

where $d\mu_C$ is a Gaussian measure on $\mathbb{R}^n$. The dependence between these random variables is entirely due to the Gaussian measure which, in general, is given by covariances $C_{ij} = \text{cov}(X_i, X_j)$ which do not vanish for $i \neq j$. A typical procedure one uses in this type of problem is to try to interpolate between the given covariance matrix $C$ and the covariance obtained by killing the off-diagonal entries. The outcome is what is called a cluster expansion in the constructive field theory literature.
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Field Theory

On forests

Tensor field theories
A new kind of analytically accessible strongly interacting fixed points
\( \varphi^4 \) MODEL IN \( d = 4 - \epsilon \)

\[
S = \frac{1}{2} \int \varphi(-\Delta + m^2)\varphi + \frac{\lambda}{4!} \int \varphi^4 \]

d < 4 cures UV divergences

subtraction scale \( \mu \) cures IR divergences.

\( g \) dimensionless effective coupling at scale \( \mu \)
**$\phi^4$ Model in $d = 4 - \epsilon$**

\[
S = \frac{1}{2} \int \varphi (-\Delta + m^2) \varphi + \frac{\lambda}{4!} \int \varphi^4
\]

- $d < 4$ cures UV divergences
- Subtraction scale $\mu$ cures IR divergences.

$g$ dimensionless effective coupling at scale $\mu$

**Beta function – scale derivative of the dimensionless effective coupling**

\[
\beta_g = \mu \partial_\mu g \bigg|_{\lambda \text{ fixed}} = -\epsilon g + \frac{3}{2} g^2 + O(g^3)
\]
The Wilson Fisher fixed point

\[ \beta_g = -\epsilon g + \beta_2 g^2 + O(g^3) \]

Stable, infrared attractive fixed point:

\[ g_* = \frac{\epsilon}{\beta_2} + O(\epsilon^2) \]
The Wilson Fisher fixed point

\[ \beta_g = -\epsilon g + \beta_2 g^2 + O(g^3) \]

Stable, infrared attractive fixed point:

\[ g_* = \frac{\epsilon}{\beta_2} + O(\epsilon^2) \]

If one only aims for the Nobel prize \( \epsilon = 1 \)

If one aims for rigour keep \( \epsilon \) small!
A TENSOR FIELD THEORY

[Carrozza Tanasa ’15, Giombi Klebanov Tarnopolsky ’16 ’17 ’18]

Rank 3 tensor \( \varphi_{b_1 b_2 b_3} = O_{b_1 a_1}^{(1)} O_{b_2 a_2}^{(2)} O_{b_3 a_3}^{(3)} \varphi_{a_1 a_2 a_3} \), invariant action

\[
S = \frac{1}{2} \int \varphi_{a_1 a_2 a_3} \left( -\Delta \right) \varphi_{a_1 a_2 a_3} + \frac{\lambda}{4N^{3/2}} \int \varphi_{a_1 a_2 a_3} \varphi_{b_1 b_2 b_3} \varphi_{c_1 c_2 c_3} \varphi_{d_1 d_2 d_3} \left\{ \delta_{a_1} b_1 \delta_{c_1} d_1 \delta_{a_2} c_2 \delta_{b_2} d_2 \delta_{a_3} b_3 \delta_{c_3} d_3 \right\}
\]
A tensor field theory

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Indices follow the strands – one sum per closed colored cycle, pairwise identifications of external indices:

\[ N^{-\frac{3}{2}} V + F \prod \delta_{a_i b_i} \]
**TWO AND FOUR POINT FUNCTIONS**

*Tetrahedron, pillow and double trace four point functions*
Formal conformal limit

Large $N$, small momentum the two point function can be obtained by solving self consistently the Schwinger Dyson equation

$$\langle \varphi(x)\varphi(y) \rangle \sim \frac{1}{|x - y|^{2d/4}}$$

suggests a non Gaussian infrared fixed point
Wilson Fisher like fixed point?

$S = \frac{1}{2} \int \varphi(-\Delta + m^2_{\text{mass}}) \varphi + \int \varphi \varphi \varphi \left( \frac{\lambda}{4N^{3/2}} \delta^t + \frac{\lambda_p}{4N^2} \delta^p + \frac{\lambda_d}{4N^3} \delta^d \right)$

4 $- \epsilon$ dimensions: fixed point $\sim \sqrt{\epsilon}$ but unstable (limit cycle)
Wilson Fisher like fixed point?

\[ S = \frac{1}{2} \int \varphi(-\Delta + \frac{m^2}{\text{mass}})\varphi + \int \varphi\varphi\varphi \left( \frac{\lambda}{4N^{3/2}} \delta^t + \frac{\lambda_p}{4N^2} \delta^p + \frac{\lambda_d}{4N^3} \delta^d \right) \]

4 \(-\epsilon\) dimensions: fixed point \(\sim \sqrt{\epsilon}\) but unstable (limit cycle)

But the tensor fixed point is in fact very different!
Conformal scaling

[Brydges Mitter Scoppola 02, Abdesselam 06]
Flow to the CFT → use form the onset the infrared scaling of the covariance

[Benedetti Gura Harribey ’19]

\[ S = \frac{1}{2} \int \varphi \left[ \left( -\Delta \right)^{\zeta = d/4} + m^2 \right] \varphi + \int \varphi \varphi \varphi \left( \frac{\lambda}{4N^{3/2}} \delta^t + \frac{\lambda_p}{4N^2} \delta^p + \frac{\lambda_d}{4N^3} \delta^d \right) \]
For $N \to \infty$, at all orders in the couplings and irrespective of the cutoff scheme the $\beta$ functions are quadratic:

$$k \partial_k g = \beta_g = 0 ,$$

$$k \partial_k g_1 = \beta_{g_1} = \beta_0^g - 2 \beta_1^g g_1 + \beta_2^g g_1^2 ,$$

$$k \partial_k g_2 = \beta_{g_2} = \beta_0^{\sqrt{3}g} - 2 \beta_1^{\sqrt{3}g} g_2 + \beta_2^{\sqrt{3}g} g_2^2 ,$$

with $\beta_0^g, \beta_1^g, \beta_2^g$ power series in the tetrahedral coupling $g$

$$g_{1 \pm} = \frac{\beta_1^g \pm \sqrt{(\beta_1^g)^2 - \beta_0^g \beta_2^g}}{\beta_2^g} = \pm i g + O(g^2) ,$$

$$\beta'_{g_1}(g_{1 \pm}) = \pm \sqrt{(\beta_1^g)^2 - \beta_0^g \beta_2^g} = \pm i g \frac{\Gamma\left(\frac{d}{4}\right)^2}{\Gamma\left(\frac{d}{2}\right)} + O(g^2)$$

Tetrahedral invariant does not have a definite sign, pillow and double trace do -- take $g = -i |g|$ !
\( g_{1^-} \) is ultraviolet attractive and strongly interacting

\( g_{1^+} \) is infrared attractive, stable and strongly interacting

Explicit renormalization group trajectory from \( g_{1^-} \) to \( g_{1^+} \)