From constructive to tensor field theory

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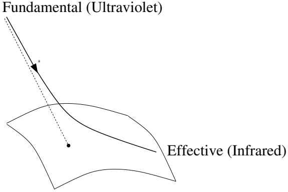
Paris, 2019

Renormalization

Physics changes with the energy scale:

Renormalization Group



Flow in the space of theories ("time" $\sim$ energy scale) [Polchinski '84 ...]

Fixed points and trajectories

Weak versus strong coupling

Weak coupling

Perturbation theory

Non perturbative?

Strong coupling

?

Jacques's laws (around 2005)

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First law

$$0 = 0$$

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Too complicated!

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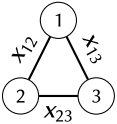
Too complicated!

Second law

The only formula you need to know is:

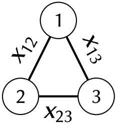
$$f(1) = f(0) + \int_0^1 dt\, f'(t)$$

For three variables...



$f(x_{12}, x_{13}, x_{23})$

For three variables...



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apply (carefully!) the second law

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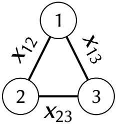
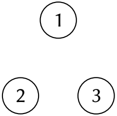
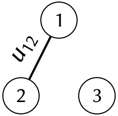


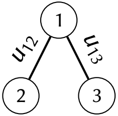
$f(x_{12}, x_{13}, x_{23})$

apply (carefully!) the second law

$$f(1,1,1) = f(0,0,0) + \int_0^1 du_{12}\, \frac{\partial f}{\partial x_{12}}(u_{12},0,0) + \dots$$
$$+ \int_0^1 du_{12}du_{13}\, \frac{\partial^2 f}{\partial x_{12}\partial x_{13}}(u_{12}, u_{13}, \inf(u_{12}, u_{13})) + \dots$$







The Brydges-Kennedy-Abdesselam-Rivasseau formula

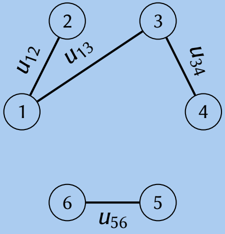
Abdesselam-Rivasseau, CPHT '94

Consider the complete graph over n vertices labelled $\{1, \dots n\}$ and let $f(x_{ij})$ be a function of the $\binom{n}{2}$ link variables x_{ij}. Then

$$f(1, \dots 1) = \sum_F \int_0^1 \left(\prod_{(k,l) \in F} du_{kl} \right) \left(\frac{\partial^{|F|} f}{\prod_{(k,l) \in F} \partial x_{kl}} \right) (w_{ij}^F) \ ,$$

- F runs over the forests (acyclic subgraphs) of the complete graph
- to each edge (k, l) in the forest we associate a variable u_{kl} which is integrated from 0 to 1
- we take the derivative of f with respect to the varibales associated to the edges in the forest
- we evaluate this derivative at $x_{ij} = w_{ij}^F$, the infimum of u along the path in F connecting the vertices i and j

The w_{ij}^F matrix



$$w^F = \begin{pmatrix} 1 & u_{12} & u_{13} & \inf(u_{13}, u_{34}) & 0 & 0 \\ \dots & 1 & \inf(u_{12}, u_{13}) & \inf(u_{12}, u_{13}, u_{34}) & 0 & 0 \\ \dots & \dots & 1 & u_{34} & 0 & 0 \\ \dots & \dots & \dots & 1 & 0 & 0 \\ \dots & \dots & \dots & \dots & 1 & u_{56} \\ \dots & \dots & \dots & \dots & \dots & 1 \end{pmatrix} \geq 0!$$

Why the BKAR formula is important

Notes on the Brydges-Kennedy-Abdesselam-Rivasseau forest interpolation formula

There are many instances in mathmatical physics where one tries to understand joint probability measures for a collection of random variables $X_1, \dots, X_n$ with n large, of the form

$$e^{-\sum_{i=1}^{n} V(x_i)} d\mu_C(x) \ ,$$

where $d\mu_C$ is a Gaussian measure on $\mathbb{R}^n$. The dependence between these random variables is entierly due to the Gaussian measure which, in general, is given by covariances $C_{ij} = \mathrm{cov}(X_i, X_j)$ which do not vanish for $i \neq j$. A typical procedure one uses in this type of problem is to try to interpolate between the given covariance matrix C and the covariance obtained by killing the off-diagonal entries. The outcome is what is called a cluster expansion in the constructive field theory literature.

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Immagine what one can do with Jacques's first law $0 = 0$!

A new kind of analytically accessible strongly interacting fixed points

φ^4 MODEL IN $d = 4 - \epsilon$

$$S = \frac{1}{2}\int \varphi(-\Delta + m^2)\varphi + \frac{\lambda}{4!}\int \varphi^4$$

$d < 4$ cures UV divergences

subtraction scale μ cures IR divergences.

g dimensionless effective coupling at scale μ

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Beta function – scale derivative of the dimensionless effective coupling

$$\beta_g = \mu\partial_\mu g\big|_{\lambda\text{ fixed}} = -\epsilon g + \frac{3}{2}g^2 + O(g^3)$$

The Wilson Fisher fixed point

$$\beta_g = -\epsilon g + \beta_2 g^2 + O(g^3)$$

Stable, infrared attractive fixed point:

$$g_\star = \frac{\epsilon}{\beta_2} + O(\epsilon^2)$$

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Stable, infrared attractive fixed point:

$$g_\star = \frac{\epsilon}{\beta_2} + O(\epsilon^2)$$

If one only aims for the Nobel prize $\epsilon = 1$

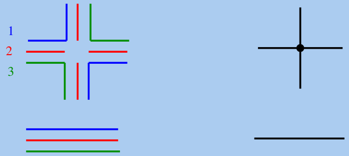
If one aims for rigour keep ϵ small!

A TENSOR FIELD THEORY

[Carrozza Tanasa '15, Giombi Klebanov Tarnopolsky '16 '17 '18]

Rank 3 tensor $\varphi_{b_1 b_2 b_3} = O^{(1)}_{b_1 a_1} O^{(2)}_{b_2 a_2} O^{(3)}_{b_3 a_3} \varphi_{a_1 a_2 a_3}$, invariant action

$$S = \frac{1}{2} \int \varphi_{a_1 a_2 a_3} (-\Delta) \varphi_{a_1 a_2 a_3} + \frac{\lambda}{4 N^{3/2}} \int \varphi_{a_1 a_2 a_3} \varphi_{b_1 b_2 b_3} \varphi_{c_1 c_2 c_3} \varphi_{d_1 d_2 d_3} \underbrace{\delta_{a_1 b_1} \delta_{c_1 d_1} \delta_{a_2 c_2} \delta_{b_2 d_2} \delta_{a_3 d_3} \delta_{b_3 c_3}}_{\delta^t}$$



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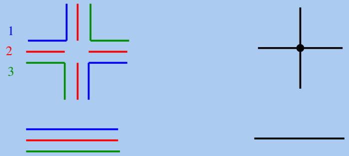
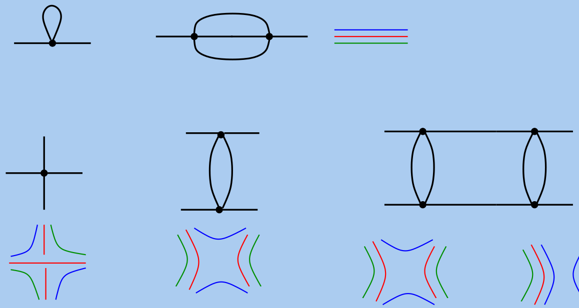
$$S = \frac{1}{2}\int \varphi_{a_1a_2a_3}(-\Delta)\varphi_{a_1a_2a_3} + \frac{\lambda}{4N^{3/2}}\int \varphi_{a_1a_2a_3}\varphi_{b_1b_2b_3}\varphi_{c_1c_2c_3}\varphi_{d_1d_2d_3} \underbrace{\delta_{a_1b_1}\delta_{c_1d_1}\delta_{a_2c_2}\delta_{b_2d_2}\delta_{a_3d_3}\delta_{b_3c_3}}_{\delta^t}$$



Indices follow the strands – one sum per closed colored cycle, pairwise identifications of external indices:

$$N^{-\frac{3}{2}V+F}\prod \delta_{a_ib_i}$$

Two and four point functions



Tetrahedron, *pillow* and *double trace* four point functions

Formal conformal limit

[Giombi Klebanov Tarnopolsky '17]

Large N, small momentum the two point function can be obtained by solving self consistently the Schwinger Dyson equation

$$\langle \varphi(x)\varphi(y)\rangle \sim \frac{1}{|x-y|^{2\frac{d}{4}}}$$

suggests a non Gaussian infrared fixed point

Wilson Fisher like fixed point?

[Giombi Klebanov Tarnopolsky '17]

$$S = \frac{1}{2}\int \varphi(-\Delta + \underbrace{m^2}_{\text{mass}})\varphi + \int \varphi\varphi\varphi\varphi\left(\frac{\lambda}{4N^{3/2}}\delta^{\mathrm{t}} + \underbrace{\frac{\lambda_p}{4N^2}\delta^{\mathrm{p}}}_{\text{pillow}} + \underbrace{\frac{\lambda_d}{4N^3}\delta^{\mathrm{d}}}_{\text{double trace}}\right)$$

$4 - \epsilon$ dimensions: fixed point $\sim \sqrt{\epsilon}$ but unstable (limit cycle)

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$4-\epsilon$ dimensions: fixed point $\sim \sqrt{\epsilon}$ but unstable (limit cycle)

But the tensor fixed point is in fact very different!

Conformal scaling

[Brydges Mitter Scoppola 02, Abdesselam 06]

Flow to the CFT $\to$ use form the onset the infrared scaling of the covariance

[Benedetti Gurau Harribey '19]

$$S = \frac{1}{2}\int \varphi\Big[\underbrace{(-\Delta)^{\zeta}}_{\zeta=d/4} + m^2\Big]\varphi + \int \varphi\varphi\varphi\varphi\left(\frac{\lambda}{4N^{3/2}}\delta^{\mathrm{t}} + \frac{\lambda_p}{4N^2}\delta^{\mathrm{p}} + \frac{\lambda_d}{4N^3}\delta^{\mathrm{d}}\right)$$

Beta functions at all orders

For $N \to \infty$, *at all orders in the couplings and irrespective* of the cutoff scheme* the β functions are quadratic:

$$
\begin{aligned}
k\partial_k g &= \beta_g = 0 \ , \\
k\partial_k g_1 &= \beta_{g_1} = \beta_0^g - 2\beta_1^g \, g_1 + \beta_2^g \, g_1^2 \ , \\
k\partial_k g_2 &= \beta_{g_2} = \beta_0^{\sqrt{3}g} - 2\beta_1^{\sqrt{3}g} \, g_2 + \beta_2^{\sqrt{3}g} \, g_2^2 \ ,
\end{aligned}
$$

with β_0^g, β_1^g, β_2^g power series in the tetrahedral coupling g

$$
g_{1\pm} = \frac{\beta_1^g \pm \sqrt{(\beta_1^g)^2 - \beta_0^g \beta_2^g}}{\beta_2^g} = \pm \mathrm{i}\, g + O(g^2) \ ,
$$

$$
\beta'_{g_1}(g_{1\pm}) = \pm\sqrt{(\beta_1^g)^2 - \beta_0^g \beta_2^g} = \pm \mathrm{i}\, g \; 4\frac{\Gamma(\frac{d}{4})^2}{\Gamma(\frac{d}{2})} + O(g^2)
$$

Tetrahedral invariant does not have a deffinite sign, pillow and double trace do – take $g = -\mathrm{i}\,|g|$!

The fixed points and the RG trajectories

g_{1-} is ultraviolet attractive and strongly interacting

g_{1+} is infrared attractive, stable and strongly interacting

Explicit renormalization group trajectory from g_{1-} to g_{1+}