Some adventures in Condensed Matter Physics

Benoît Douçot

March 21, 2019

My interactions with CPHT (I)

A seminar by B. Souillard at CPHT in March 1984 on a rigorous proof of Anderson localization $\rightarrow$ choice of my research field !

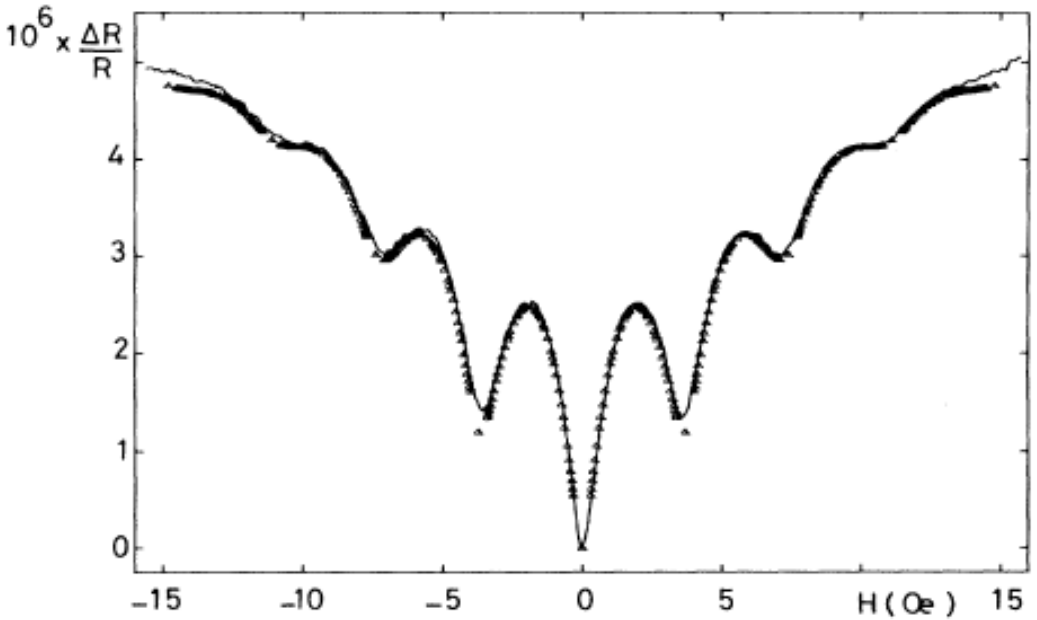
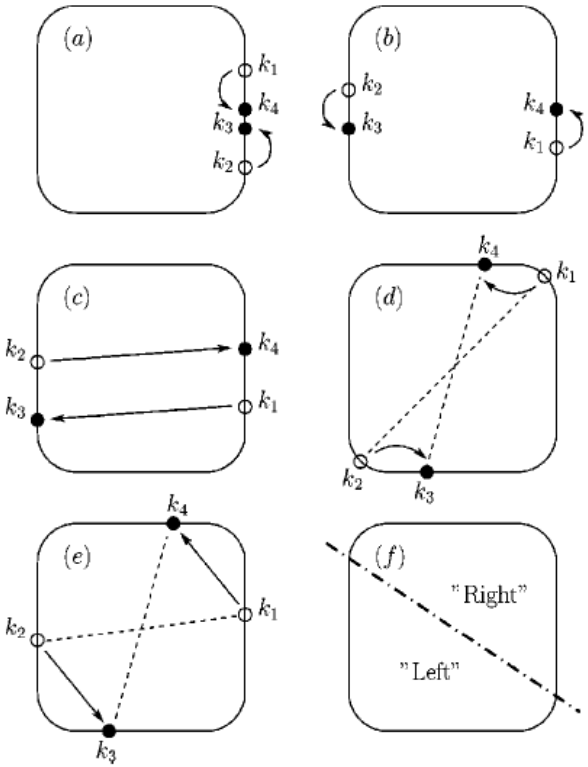


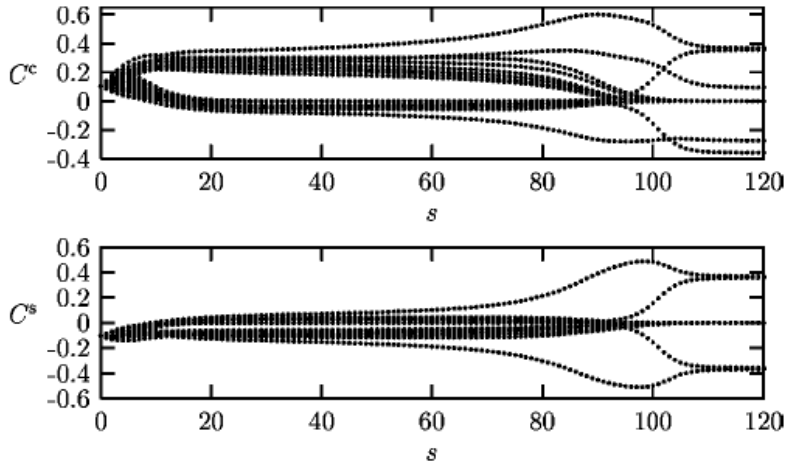
FIG. 2. Quantitative comparison between the theoretical results (triangles) and experimental data (solid line), for Cu at $T = 133$ mK, taken from Ref. 2. The hexagonal elementary cells (side $a = 1.5$ μm) are made of wires of width 0.42 μm. In this fit, we have $L\varphi = 5.36$ and $L_{s.o.} = 3.12$ μm, respectively ($L_{s.o.}$ is the spin-orbit length).

B. D. and R. Rammal, PRL **55**, 1148 (1985).

My interactions with CPHT (II)

Many discussions with V. Rivasseau, C. Kopper, M. Disertori, around 2000 helped me to understand better some aspects of RG applied to fermionic systems.

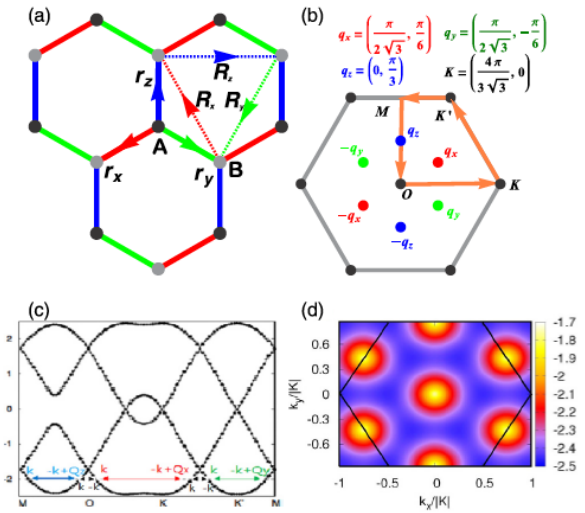


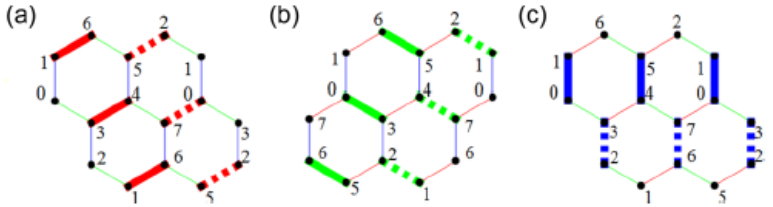


S. Dusuel, F. Vistulo de Abreu, B. D., PRB **65**, 94505 (2002).

My interactions with CPHT (III)

Co-supervision of Tianhan Liu's thesis with Karyn Le Hur.



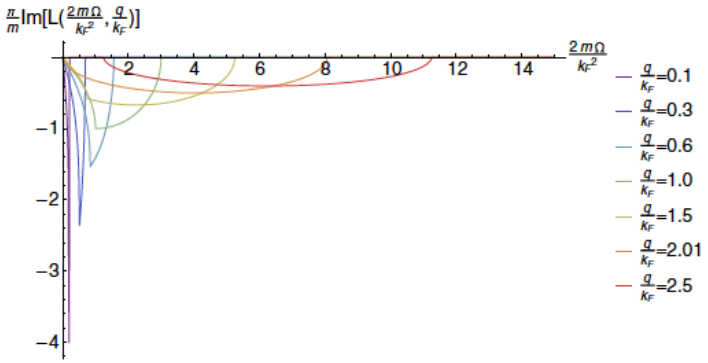


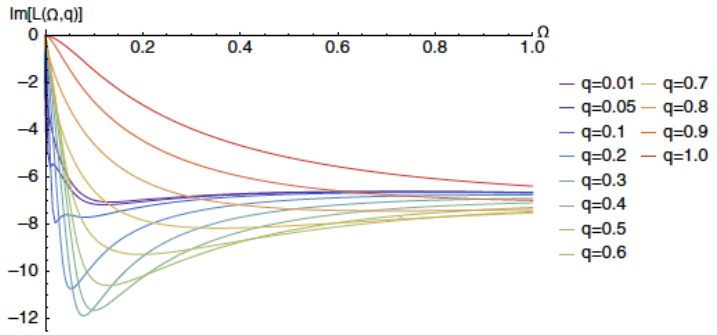
Tianhan Liu, Cécile Repellin, B.D., N. Regnault, K. Le Hur, PRB **94**, 180506 (2016).

My interactions with CPHT (IV)

Collaboration with A. Mukhopadhyay, G. Policastro on semi-holographic non-Fermi liquids.

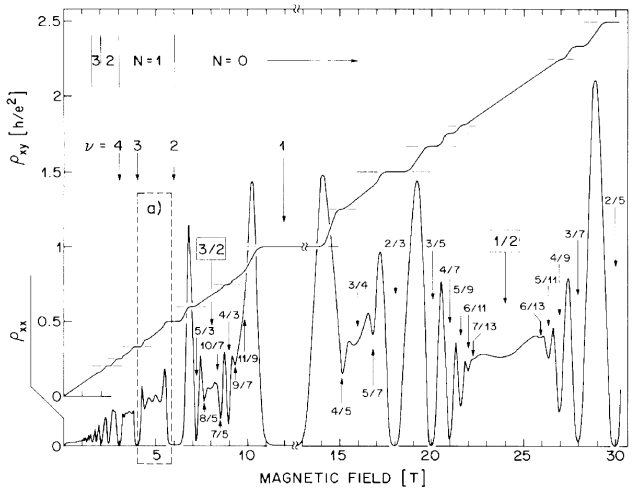
$$G(k,\omega)^{-1} = \zeta\omega^{\nu} + \omega - \epsilon(k), \quad 0 < \nu < 1$$

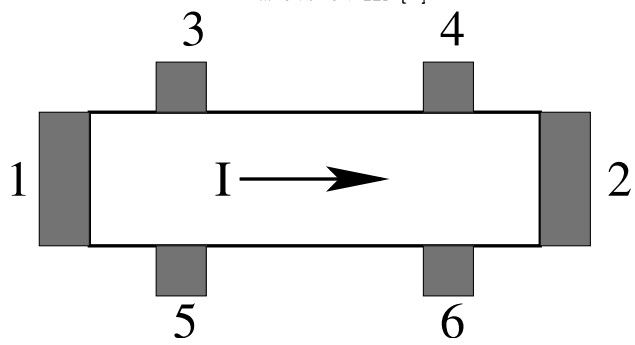




B. D., C. Ecker, A. Mukhopadhyay, G. Policastro, PRD **96**, 106011 (2017).

Quantum Hall effect





$$R_{xx} = (V(3) - V(4))/I$$
$$R_{xy} = (V(3) - V(5))/I$$

Quantum nature of Hall resistance plateaus

Plateaus observed for (ν integer):

$$\rho_{xy} = \frac{B}{ne} = \frac{h}{\nu e^2}$$

$\rightarrow$ Quantized electronic densities:

$$n = \nu \frac{eB}{h}$$

In terms of $\Phi_0 = \frac{h}{e}$: "Flux quantum"

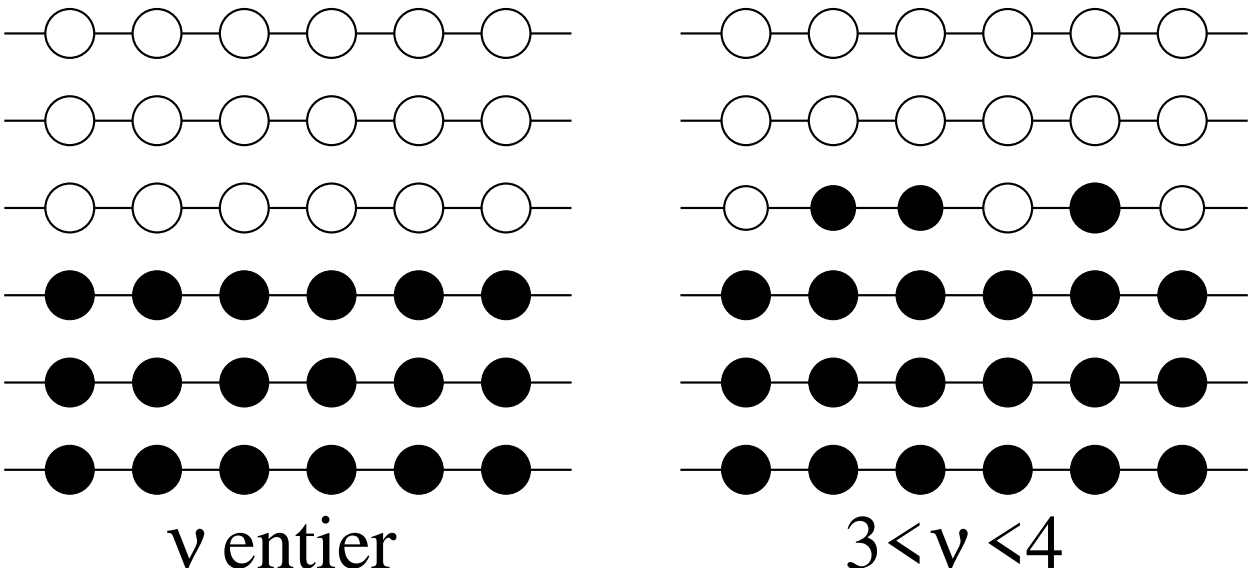
$$N_{\text{electrons}} = \nu \frac{\text{Total magnetic flux}}{\Phi_0}$$

Landau levels are degenerate

Intuitively, each state occupies the same area as a flux quantum Φ_0, so that the number of states per Landau level =

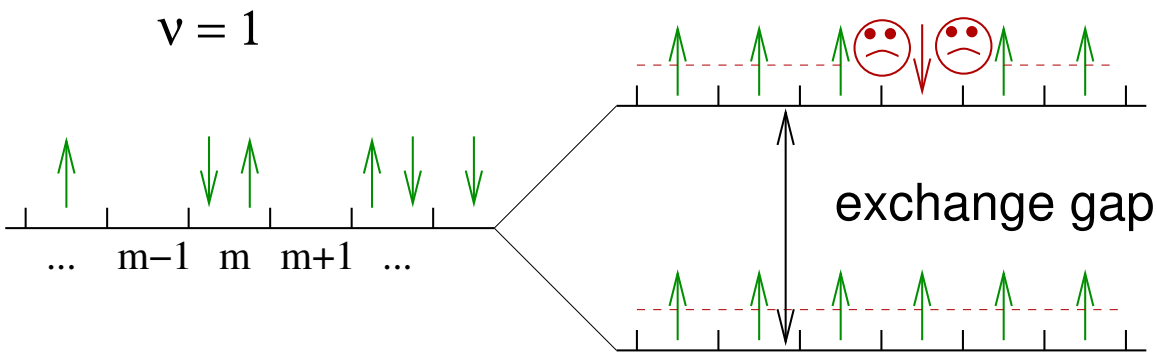
$$\frac{\text{Total magnetic flux}}{\Phi_0}$$

ν is interpreted as the number of occupied Landau levels



Ferromagnetism at $\nu = 1$

Coulomb repulsion favours anti-symmetric orbital wavefunction $\rightarrow$ spin wavefunction is symmetric $\rightarrow$ ferromagnetic state

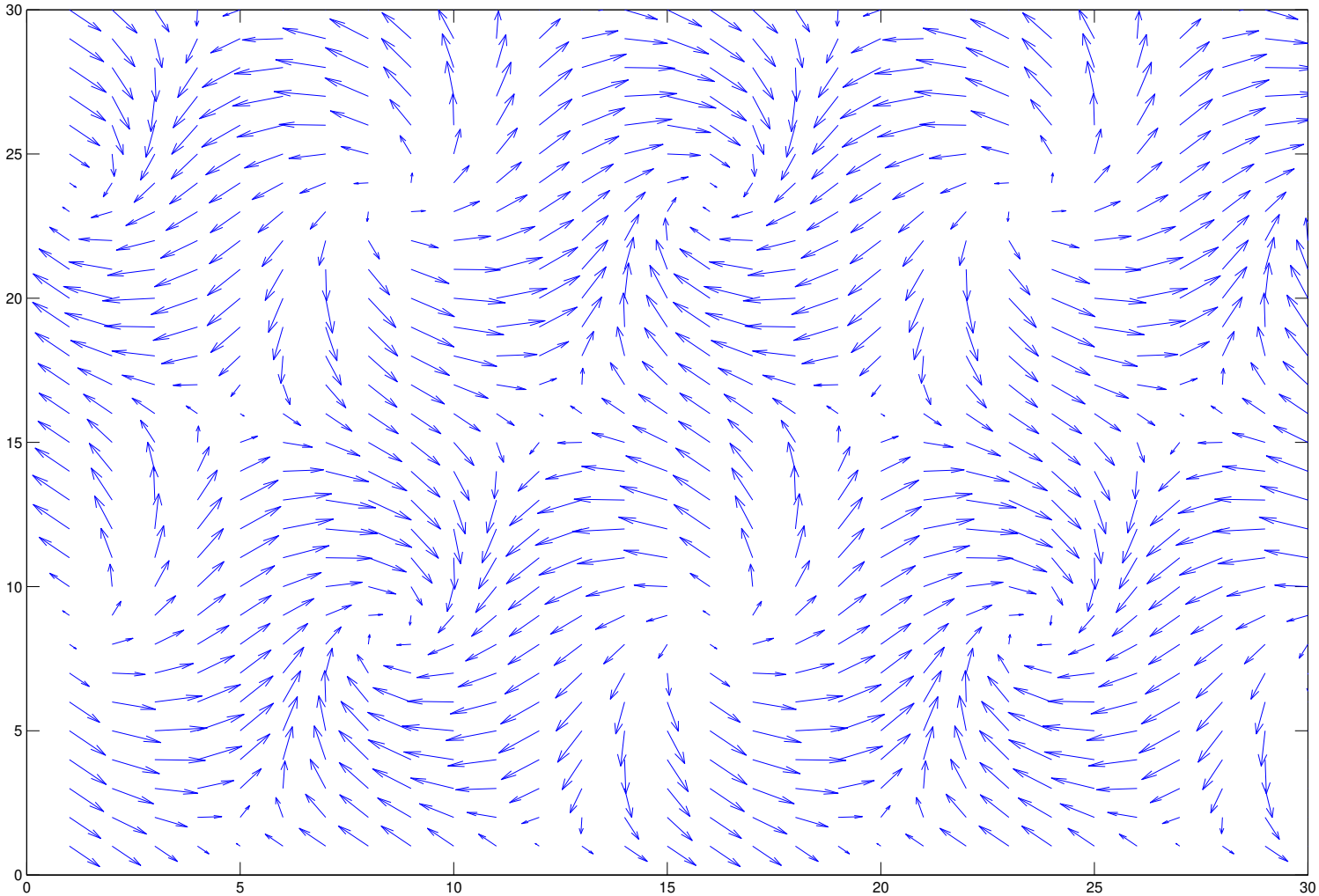


no interactions with repulsive interactions

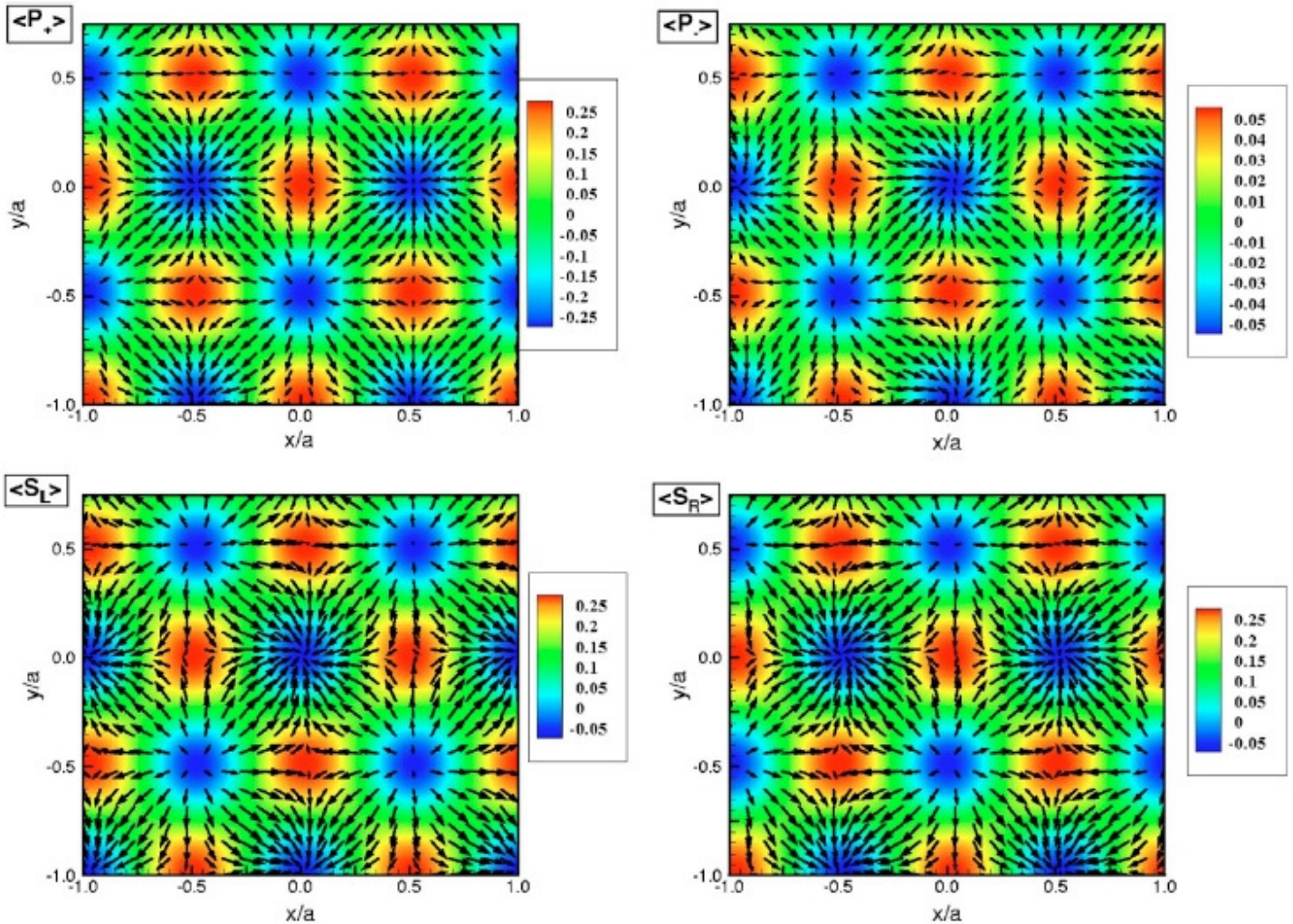
in QH systems : no kinetic–energy cost !

(LL ~ flat band)

Picture of a Skyrmion crystal



Example of entangled textures



Bourassa et al, Phys. Rev. B **74**, 195320 (2006)

A class of trial states near $\nu = 1$

Take antisymmetrized products of single particle states (Slater determinants or Hartree-Fock states): $|S_\psi\rangle = \bigwedge_{\alpha=1}^{N} |\Phi_\alpha\rangle$
where $\Phi_{\alpha,a}(\mathbf{r}) = \chi_\alpha(\mathbf{r})\psi_a(\mathbf{r})$, $\mathbf{r} = (x, y)$, $a \in \{1, ..., N\}$.
$\chi_\alpha(\mathbf{r}) \rightarrow$ electron position.
$\psi_a(\mathbf{r}) \rightarrow$ slowly varying spin background. $(\langle\psi(\mathbf{r})|\psi(\mathbf{r})\rangle = 1)$.

In the $N = 2$ case, if σ_a denote Pauli matrices:
Associated classical spin field: $n_a(\mathbf{r}) = \langle\psi(\mathbf{r})|\sigma_a|\psi(\mathbf{r})\rangle$
Topological charge: $N_{\text{top}} = \frac{1}{4\pi}\int d^{(2)}\mathbf{r}\,(\partial_x\vec{n} \wedge \partial_y\vec{n}) \cdot \vec{n}$

Because of large magnetic field, we impose that orbital wave-functions $\Phi_{\alpha,a}(\mathbf{r})$ lie in the lowest Landau level.

Questions

To each classical spinor field configuration $\psi_a(\mathbf{r})$, we associate a many electron wave-function (Slater determinant) $|S_\psi\rangle$. After projection onto the lowest Landau level, H contains only the Coulomb interaction!

1) Construction of optimal textures: how to minimize $\langle S_\psi|H|S_\psi\rangle$?

2) Absence of quantum fluctuations: $|S_\psi\rangle$ eigenstate of H ?

Energy functional

Main effect of $\mathcal{P}_{LLL}$: (Moon et al. (1995), Pasquier (2000),...)

$$\begin{aligned} n_{\rm el}(\mathbf{r}) &= \frac{1}{2\pi l^2} - Q(\mathbf{r}) + O(l^2) \\ N_{\rm el} &= N_\Phi - N_{\rm top} \quad \rightarrow \quad \text{CONSTRAINT} \end{aligned}$$

Energy functional:

$$E_{{\rm tot},\psi} \equiv \langle S_\psi | H | S_\psi \rangle = E_{{\rm loc},\psi} + E_{{\rm non-loc},\psi}$$

$E_{{\rm loc},\psi}$: exchange energy (generalized ferromagnet), given by a non-linear σ model energy functional (next slide).

$$E_{{\rm non-loc},\psi} = \frac{e^2}{8\pi\epsilon} \int d^2\mathbf{r} \int d^2\mathbf{r}' \, \frac{Q(\mathbf{r})Q(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

$E_{\rm loc}$ as $\mathbb{C}P(N-1)$ model for exchange energy

Assume $SU(N)$ *global* symmetry and *local* gauge symmetry: $|\psi(r)\rangle \to e^{i\phi(r)}|\psi(r)\rangle$.

$$E_{\rm loc} = \int d^{(2)}\mathbf{r} \left(\frac{\langle \nabla\psi|\nabla\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\nabla\psi|\psi\rangle\langle\psi|\nabla\psi\rangle}{\langle\psi|\psi\rangle^2} \right)$$

Berry connection: $\mathcal{A} = \frac{1}{i}\langle\psi|\nabla\psi\rangle$
Topological charge: $\oint \mathcal{A}.d\mathbf{r} = 2\pi N_{\rm top}$

$$E_{\rm loc} \geq \pi |N_{\rm top}|$$

Lower bound is reached when $|\psi(r)\rangle$ is holomorphic or anti-holomorphic: leading to a huge degeneracy.

Strategy to minimize energy functional

$$E_{\mathrm{tot},\psi} = E_{\mathrm{loc},\psi} + E_{\mathrm{non-loc},\psi}$$

If filling factor is close to M, $E_{\mathrm{non-loc}} << E_{\mathrm{loc}}$. To find optimal textures, we can therefore:

1. Minimize E_{loc} in the presence of the $N_{\mathrm{el}} = N_{\Phi} - Q_{\mathrm{top}}$ constraint. This leads to a continuous family of degenerate configurations.
2. Lift this degeneracy by minimizing $E_{\mathrm{non-loc}}$ within this degenerate family. Physically, this favors textures in which the topological charge density is as uniform as possible.

Holomorphic maps from the sphere to $\mathbb{C}\mathrm{P}(N-1)$ (I)

$S^2 \cong \mathbb{C}\mathrm{P}(1) \cong \mathbb{C} \cup \{\infty\}$ so we use one coordinate $z \in \mathbb{C}$.
Holomorphic maps $f : S^2 \to \mathbb{C}\mathrm{P}(N-1)$: collections of N polynomials $P_1(z), ..., P_N(z)$.
Topological charge: number of intersection points of $f(S^2)$ with an arbitrary hyperplane in $\mathbb{C}\mathrm{P}(N-1)$ = maximal degree d of $P_1(z), ..., P_N(z)$.
Topological charge density:

$$Q(z, \bar{z}) = (1 + |z|^2)^2 \partial_z \partial_{\bar{z}} \log(\sum_{i=1}^{N} |P_i(z)|^2)$$

$Q(z, \bar{z})$ is constant when:

$$\sum_{i=1}^{N} |P_i(z)|^2 = (1 + |z|^2)^d$$

Holomorphic maps from the sphere to $\mathbb{C}\mathrm{P}(N-1)$ (II)

Hermitian scalar product on degree d polynomials:

$$(P,Q)_d = \frac{d+1}{\pi}\int d^2\mathbf{r}\,\frac{\overline{P(z)}Q(z)}{(1+|z|^2)^{d+2}}$$

Orthonormal basis: $e_p(z) = \begin{pmatrix} d \\ p \end{pmatrix}^{1/2} z^p$

General texture of degree d: $P_j(z) = \sum_{i=0}^{d} A_{ij} e_i(z)$

$Q(z,\bar{z})$ is constant when raws of A are orthonormal.

If $d \geq N$: No solution.

If $d \leq N-2$: many solutions, but not all components of the maps are linearly independent.

If $d = N-1$: $AA^\dagger = I_N = A^\dagger A$, so $(P_i, P_j)_d = \delta_{ij}$.

Textures with uniform topological charge density $\Leftrightarrow$ Components form an orthonormal basis.

Holomorphic maps from the torus to $\mathbb{C}P(N-1)$ (I)

$$\theta(z+\gamma) = e^{a_\gamma z + b_\gamma}\theta(z)$$

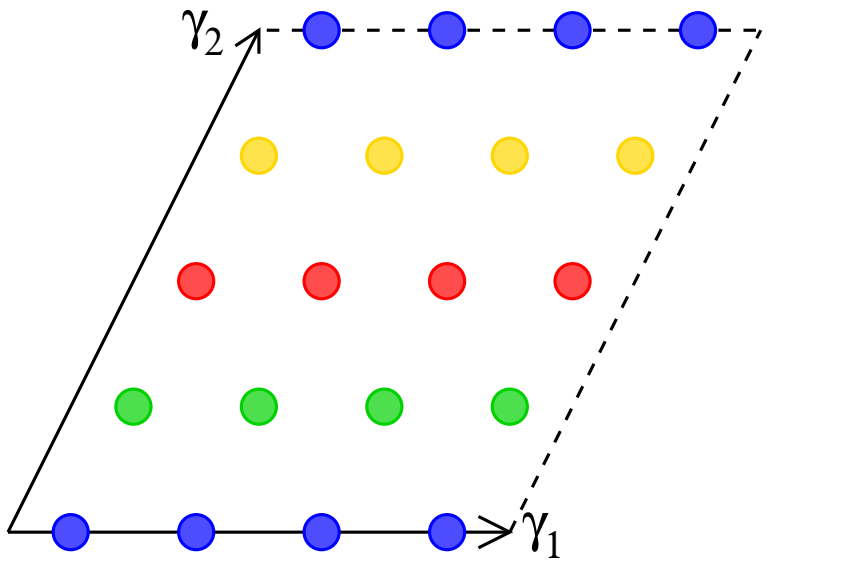
$$(\theta,\theta')_d = \int d^2\mathbf{r}\ \exp\left(-\frac{\pi d|z|^2}{|\gamma_1 \wedge \gamma_2|}\right)\overline{\theta(z)}\theta'(z)$$

Optimal textures ($d = N$)

$$|\Psi(z)\rangle = \begin{pmatrix} \theta_0(z) \\ \theta_1(z) \\ . \\ . \\ . \\ \theta_{d-1}(z) \end{pmatrix}$$

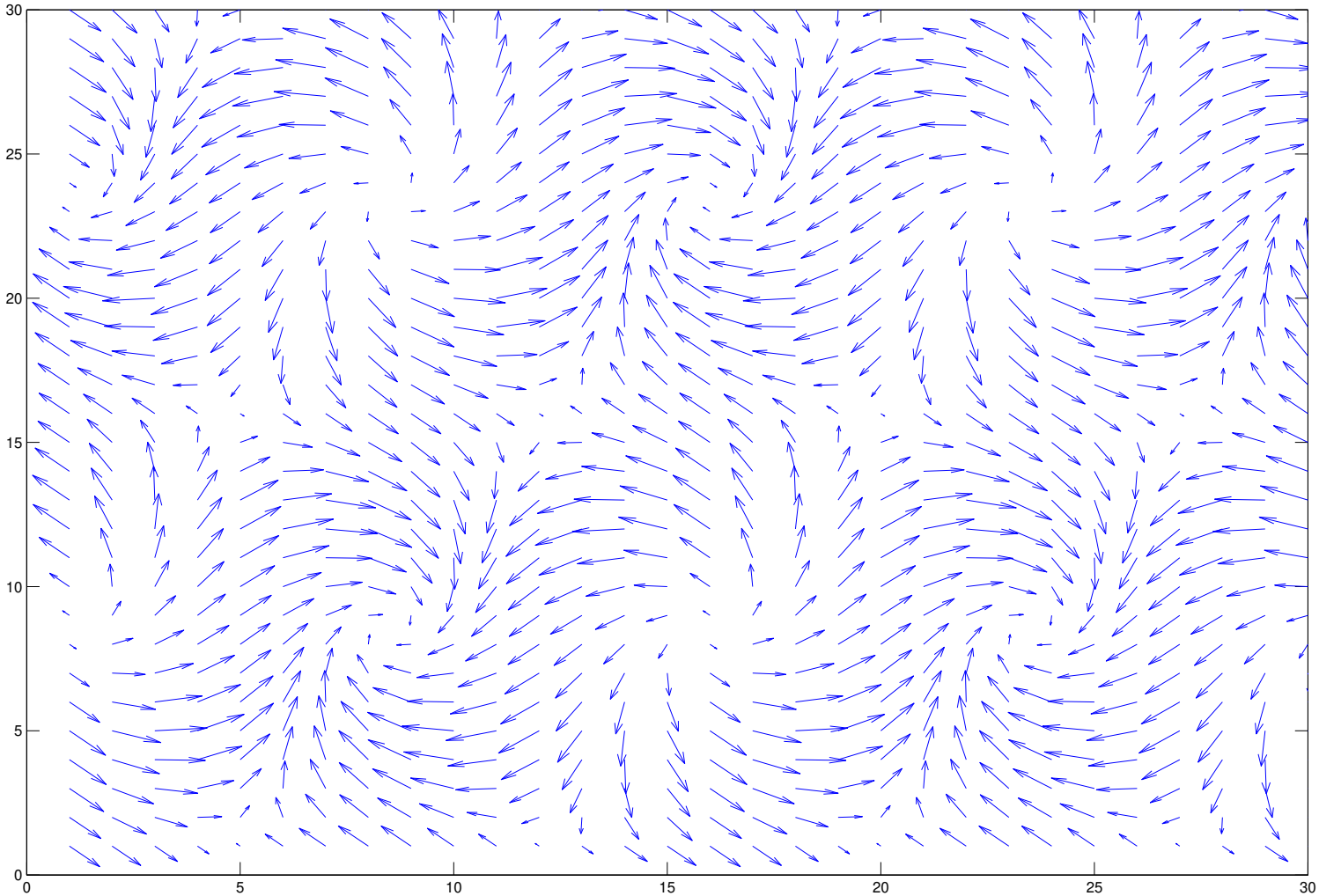
$$(\theta_i,\theta_j)_d = \delta_{ij}$$

Pattern of zeros (d=4)



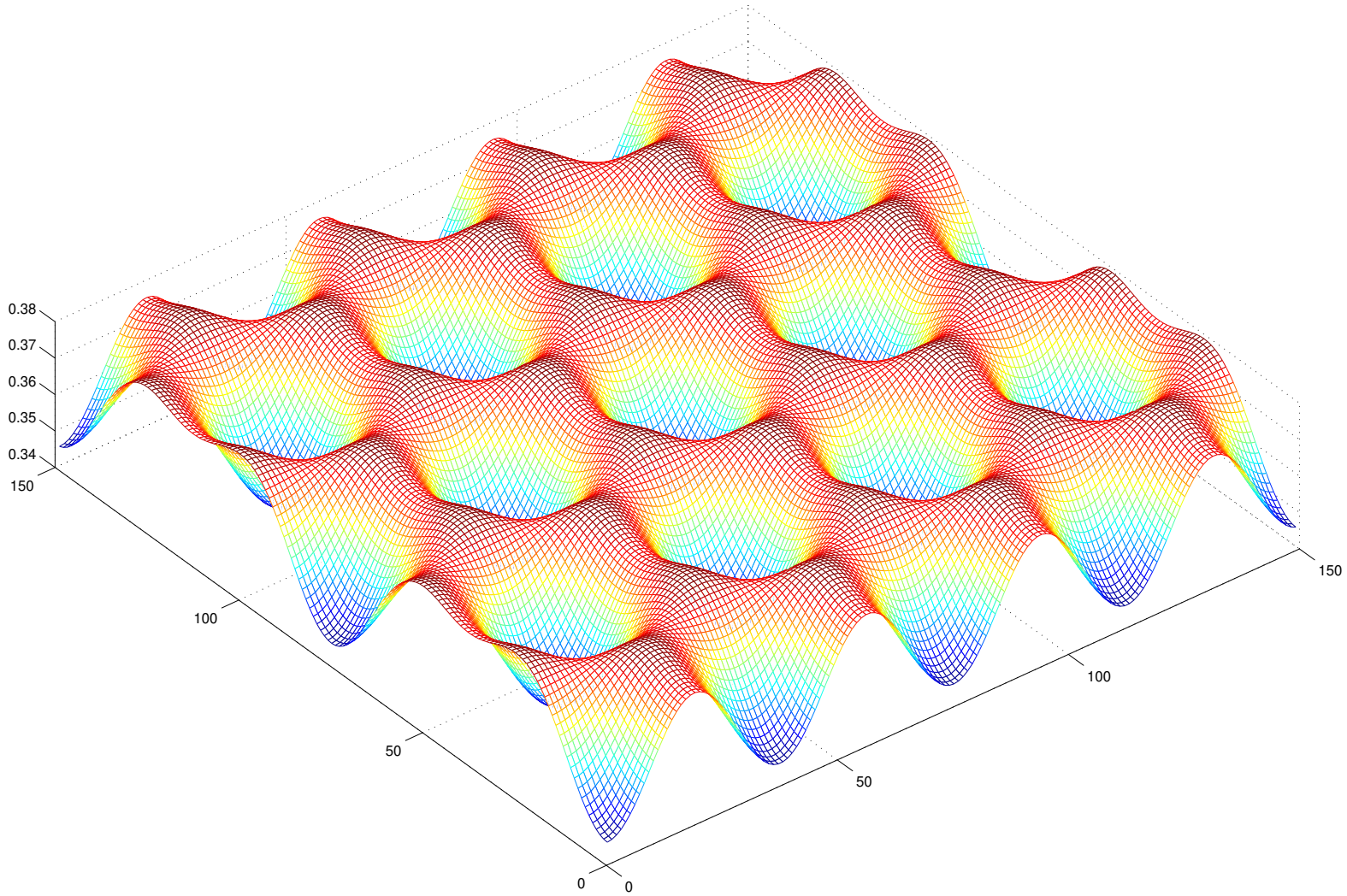
Holomorphic maps from the torus to $\mathbb{C}\mathrm{P}(N-1)$ (II)

$$d = N = 2$$



Holomorphic maps from the torus to $\mathbb{C}\mathrm{P}(N-1)$ (III)

$$d = N = 4$$



Holomorphic maps from the torus to $\mathbb{C}P(N-1)$ (IV)

Spatial variations of topological charge: $Q(r)$ is always γ_1/d and γ_2/d periodic. Unlike on the sphere, $Q(r)$ is not exactly constant.

At large d the modulation contains mostly the lowest harmonic, and its amplitude decays exponentially with d.

Large d behavior for a square lattice:

$$Q(x,y) \simeq \frac{2}{\pi} - 4de^{-\pi d/2}[\cos(2\sqrt{d}x) - 2e^{-\pi d/2}\cos^2(4\sqrt{d}x) + (x \leftrightarrow y)] + \ldots$$

Only the triangular lattice seems to yield a true local energy minimum. This has been evidenced by computing eigenfrequencies of small deformation modes.

B. Douçot, D. Kovrizhin, R. Moessner, PRL 110, 186802 (2013)

Holomorphic maps from Σ to $\mathbb{C}\mathrm{P}(N-1)$ (I)

Components of a map $f : \Sigma \to \mathbb{C}\mathrm{P}(N-1)$ were polynomials on the sphere and θ functions on the torus. Note that polynomials have poles at $z \to \infty$, and θ functions are multivalued.

More general construction: Pick a line bundle L over Σ, and choose the components of the maps $s_j(z)$ as global holomorphic sections of L, for $1 \leq j \leq N$.

Recipe for optimal textures: $N =$ dimension of the space of global holomorphic sections of L. Choose components forming an orthonormal basis for a well chosen hermitian product.

Holomorphic maps from Σ to $\mathbb{C}\mathrm{P}(N-1)$ (II)

Geometric quantization recipe for the hermitian product

ω : volume form associated to constant curvature metric on Σ

h^d : hermitian metric on fibers of L^d whose curvature form equals $-d(2\pi i)\omega$

$$(s, s')_{L,d} = \int_\Sigma h^d(s(x), s'(x))\omega(x)$$

Topological charge form: $\omega_{\mathrm{top}} - \omega = \frac{1}{\pi}\partial_z\partial_{\bar{z}} \log B(z, \bar{z})$.

$B(z, \bar{z})_{L,d} = \sum_{j=1}^N h^d(s_j(z), s_j(z))$

For an orthonormal basis $B(z, \bar{z})$ is the Bergman kernel, whose large d asymptotics has been studied a lot in the 90's.

Holomorphic maps from Σ to $\mathbb{C}P(N-1)$ (III)

Bergman kernel asymptotics (Tian, Yau, Zelditch, Catlin, Lu,...(1990 to 2000)):
$B(z,\bar{z}) = d + a_0(z,\bar{z}) + a_{-1}(z,\bar{z})d^{-1} + a_{-2}(z,\bar{z})d^{-2} + ...$, such that $a_j(z,\bar{z})$ is a polynomial in the curvature and its covariant derivatives at $(z,\bar{z})$.

Interesting consequence: If ω is associated to the constant curvature metric on Σ, the previous family of textures have uniform topological charge, up to corrections which are smaller than any power of $1/d$.

Practical question: How to effectively construct such orthonormal bases of sections, when Σ has genus ≥ 2 ?

Questions

To each classical spinor field configuration $\psi_a(\mathbf{r})$, we associate a many electron wave-function (Slater determinant) $|S_\psi\rangle$. After projection onto the lowest Landau level, H contains only the Coulomb interaction!

1) Construction of optimal textures: how to minimize $\langle S_\psi|H|S_\psi\rangle$?

2) Absence of quantum fluctuations: $|S_\psi\rangle$ eigenstate of H ?

Quantum degeneracy among holomorphic textures

First step: View Slater determinants $|S_\psi\rangle$ as coherent states.

Question: Suppose the energy functional $E_\psi = \langle S_\psi|H|S_\psi\rangle$ has a local minimum at $\psi = \psi_0$. Can we know if $|S_{\psi_0}\rangle$ is also an eigenstate of H ?

Toy model: Anharmonic oscillator $[b, b^\dagger] = 1$. Take

$$\hat{H} = E_0 + \hbar\omega_0\, b^\dagger b + \frac{\hbar\Delta}{2}(b^\dagger)^2 + \frac{\hbar\bar{\Delta}}{2} b^2 + ...$$

Pick coherent states $|\Phi_{\bar{z}}\rangle = e^{-\frac{|z|^2}{2}} e^{\bar{z}b^\dagger}|0\rangle$ to define energy functional $E(z,\bar{z}) \equiv \langle\Phi_{\bar{z}}|\hat{H}|\Phi_{\bar{z}}\rangle$, which has a local minimum at $z = 0$. In fact:

$$E(z,\bar{z}) = E_0 + \frac{\omega_0}{2}\bar{z}z + \frac{\Delta}{4}z^2 + \frac{\bar{\Delta}}{4}\bar{z}^2 + ...$$

$|0\rangle$ eigenstate of $\hat{H} \Leftrightarrow$ pure monomials $(b^\dagger)^n$ and b^n do not appear in Wick-ordered expansion of $\hat{H} \Leftrightarrow$ pure monomials z^n and $\bar{z}^n$ do not appear in Taylor expansion of $E(z,\bar{z})$.

Application to $\mathbb{C}\mathrm{P}(N-1)$ energy functional

First replace z by the classical field $|\psi(\mathbf{r})\rangle$.

$$E_{\mathrm{loc}} = \pi |N_{\mathrm{top}}| + 2 \int d^{(2)}\mathbf{r} \left(\frac{\langle \partial_z \psi | \partial_{\bar{z}} \psi \rangle}{\langle \psi | \psi \rangle} - \frac{\langle \partial_z \psi | \psi \rangle \langle \psi | \partial_{\bar{z}} \psi \rangle}{\langle \psi | \psi \rangle^2} \right)$$

Pick a holomorphic texture $|\psi_0\rangle$ and expand $|\psi(\mathbf{r})\rangle = |\psi_0(\mathbf{r})\rangle + |\chi(\mathbf{r})\rangle$.

$\partial_{\bar{z}} |\psi_0\rangle = 0 \Rightarrow$ the Taylor expansion of E_{loc} around $|\psi_0\rangle$ does not contain any term involving only $\chi_a(\mathbf{r})$'s nor any term involving only $\bar{\chi}_a(\mathbf{r})$'s.

B. Douçot, D. Kovrizhin, R. Moessner, PRB 93, 094426 (2016)

Coherent states on $\mathbb{C}P(d-1)$ (I)

Consider d bosonic modes with $[a_i, a_j^\dagger] = \delta_{ij}$ for $0 \leq i, j \leq d-1$. Take m positive integer, and consider the finite dimensional subspace of bosonic Fock space, defined by the constraint: $\sum_{i=0}^{N-1} a_i^+ a_i = m$.

Orthonormal basis: $|\vec{n}\rangle = \frac{(\hat{a}_0^+)^{n_0} \cdots (\hat{a}_{N-1}^+)^{n_{N-1}}}{\sqrt{n_0! \cdots n_{N-1}!}} |0\rangle, \quad \sum_{i=0}^{N-1} n_i = m,$

$n_i \geq 0$.

Coherent states: $|e_{\bar{v}}\rangle = \sum \frac{\bar{v}_1^{n_1} \cdots \bar{v}_{N-1}^{n_{N-1}}}{\sqrt{n_0! \cdots n_{N-1}!}} |\vec{n}\rangle$

Overlaps: $\langle e_{\bar{v}'} | e_{\bar{v}} \rangle = \frac{(1 + \langle v | v' \rangle)^m}{m!}$

Reproducing kernel: $\mathbf{I} = \frac{(m+N-1)!}{\pi^{(N-1)} m!} \int \frac{\prod_{j=1}^{N-1} dv_j d\bar{v}_j}{(1+\langle v|v\rangle)^N} \frac{|e_{\bar{v}}\rangle\langle e_{\bar{v}}|}{\langle e_{\bar{v}} | e_{\bar{v}} \rangle}$

Coherent states on $\mathbb{C}P(d-1)$ (II)

Classical functional: $\frac{\langle e_{\bar{v}}| \prod_{j=0}^{N-1} (a_j^+)^{m_j} a_j^{n_j} |e_{\bar{v}}\rangle}{\langle e_{\bar{v}}|e_{\bar{v}}\rangle} = \frac{m!}{(m-n)!} \frac{\prod_{j=1}^{N-1} v_j^{m_j} \bar{v}_j^{n_j}}{(1+\langle v|v\rangle)^n}$

Consider $\hat{H}$ an operator which can be written as a power series in bosonic mode operators a_i, $a_j^\dagger$, and whose associated functional $E(v,\bar{v}) = \frac{\langle e_{\bar{v}}|\hat{H}|e_{\bar{v}}\rangle}{\langle e_{\bar{v}}|e_{\bar{v}}\rangle}$ is such that $E(v,\bar{v})$ has a minimum at $v=0$ and its Taylor expansion around $v=0$ doesn't contain any monomial composed only of v_j's nor only of $\bar{v}_j$'s.

Then $|e_0\rangle$ is an exact eigenstate of $\hat{H}$, with eigenvalue $E(0,0)$.

Towards continuum limit: lattice regularization

Take a 2D lattice, and associate to each site the quantized Hilbert space obtained from the classical $\mathbb{C}\mathrm{P}(d-1)$ manifold, with the same m at each site. The classical limit is obtained as $m \to \infty$. Consider the Hamiltonian:

$$\hat{H} = -\sum_{\langle \mathbf{r}\mathbf{r}' \rangle} \sum_{ij} a_i^\dagger(\mathbf{r}) a_j(\mathbf{r}) a_j^\dagger(\mathbf{r}') a_i(\mathbf{r}')$$

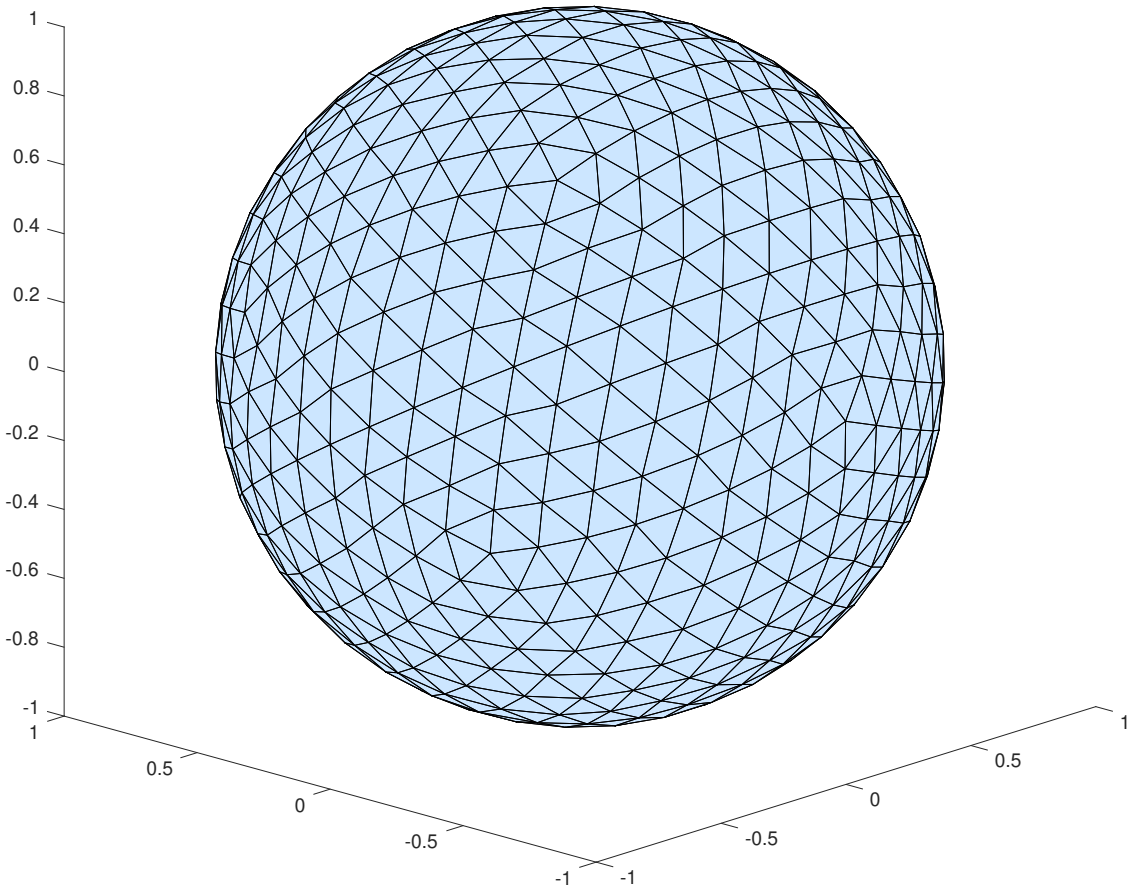
Its associated energy functional is:

$$E(v, \bar{v}) = -m^2 \frac{(1 + \langle v(\mathbf{r})|v(\mathbf{r}')\rangle)(1 + \langle v(\mathbf{r}')|v(\mathbf{r})\rangle)}{(1 + \langle v(\mathbf{r})|v(\mathbf{r})\rangle)(1 + \langle v(\mathbf{r}')|v(\mathbf{r}')\rangle)}$$

This provides a lattice discretization of the classical $\mathbb{C}\mathrm{P}(d-1)$ energy functional, together with a well defined quantization associated to it.

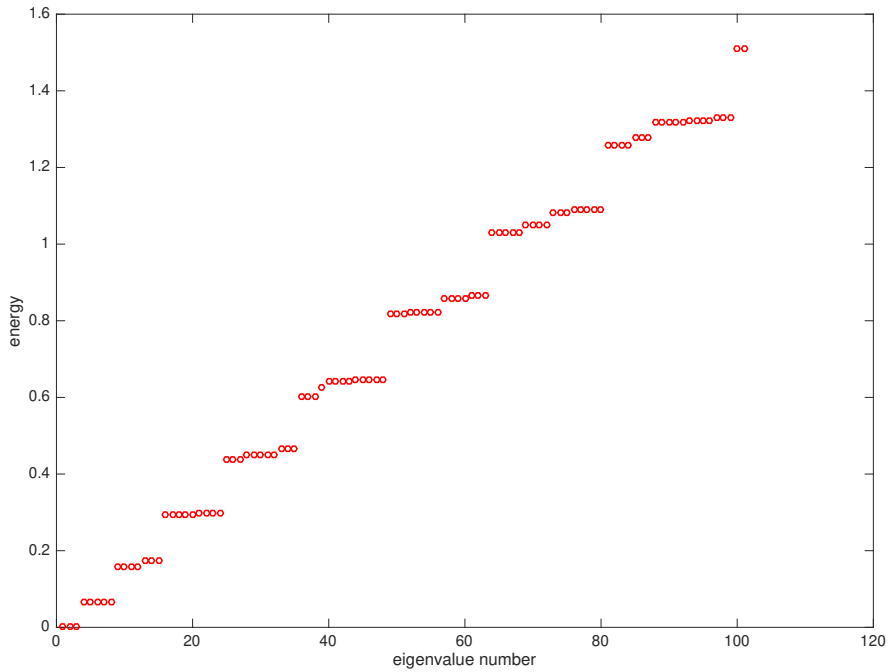
Numerical experiments (D. Kovrizhin)

Triangulation on the sphere (642 sites)



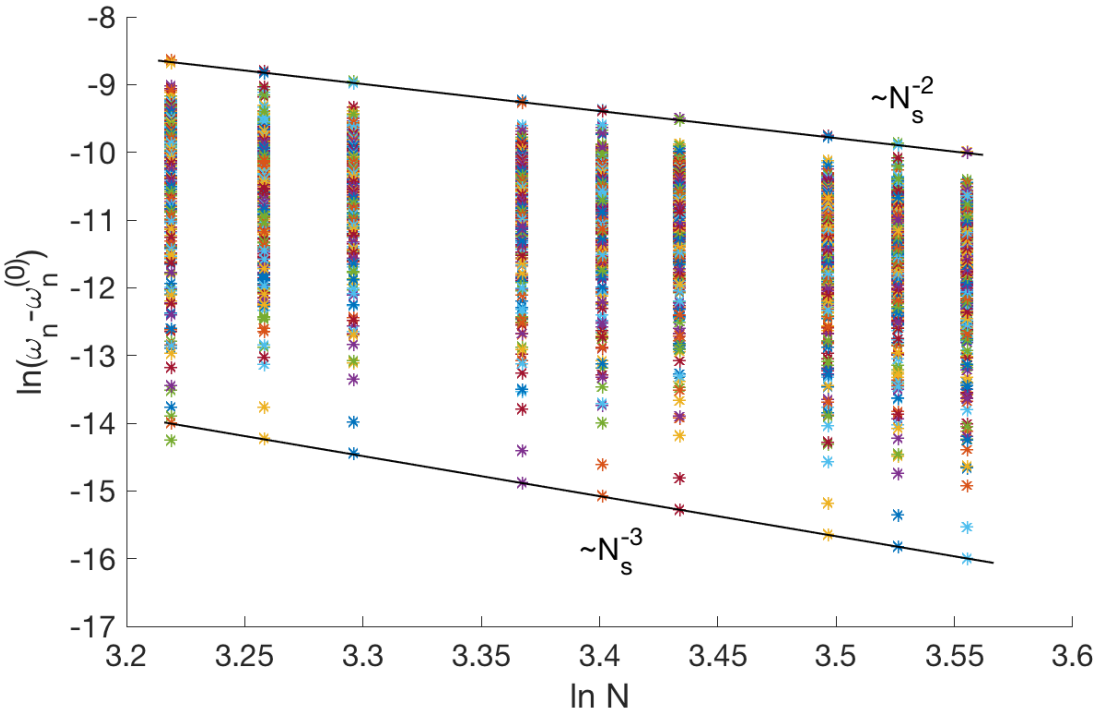
Numerical experiments (D. Kovrizhin)

Harmonic mode spectrum around a single Skyrmion classical configuration: compatible with the magnetic Laplacian on the sphere with a charge 2 magnetic monopole: manifestation of the spin Berry phase associated to a slow twist of the spin background.



Lattice effects on magnon spectrum

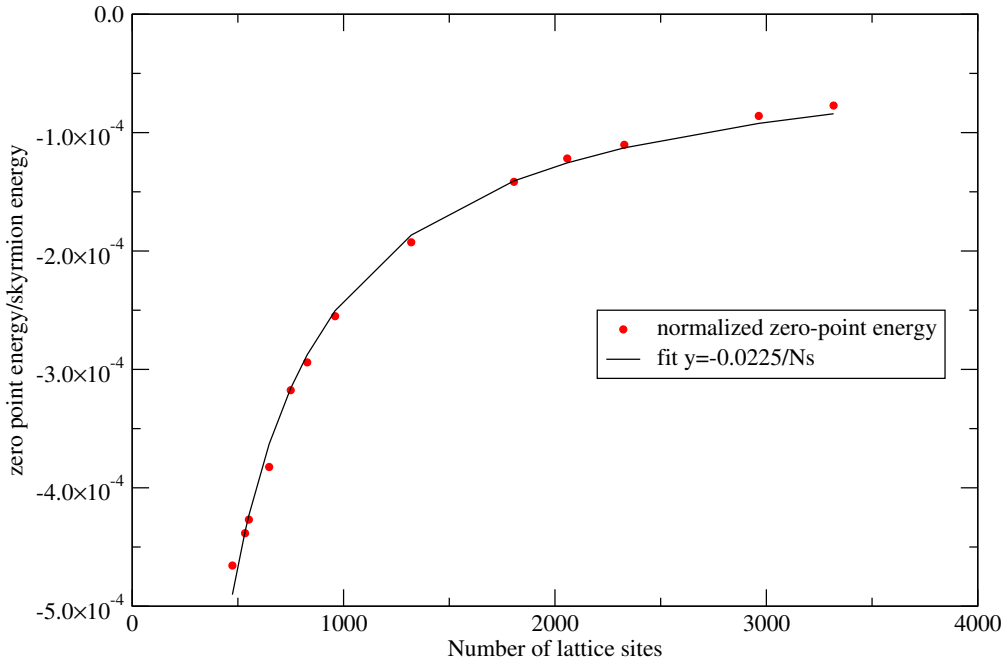
Absence of magnon-type excitations, due to holomorphic nature of the texture, holds only when magnon wave-length » lattice spacing.



Magnon frequency shift due to magnon non-conserving terms in the quadratic approximation around a classical Skyrmion solution

Quantum corrections to Skyrmion energy (I)

Harmonic mode spectrum around a single Skyrmion classical configuration gives quantum zero point correction normalized to classical Skyrmion energy



Quantum corrections to Skyrmion energy (II)

It turns out that the total quantum correction to the ground-state energy of a Skyrmion configuration goes to zero as $1/N_{\mathrm{sites}}$ when $N_{\mathrm{sites}} \to \infty$, even in the presence of small residual quantum fluctuations induced by the lattice discretization.

B. Douçot, D. Kovrizhin, R. Moessner, Annals of Physics **399**, 239 (2018).

Final remarks

Ubiquity of geometric quantization:

- Derivation of energy functionals and physical effects due to projection onto lowest Landau level.
- "Re-quantization" around classical textures and analysis of quantum zero point motion correction to total energy.
- More surprinsingly, provides a geometrical description of optimal textures, i.e. those with most uniform topological charge density.

A few open questions

- To identify optimal textures for filling factor ν close to an integer $M \geq 2$. For undoped graphene in a magnetic field, we have $N = 4$ internal states, and $M = 2$. Idea: take Slater determinants with M occupied states at a given position $\mathbf{r}$, or equivalently, an M-dimensional subspace in $\mathbb{C}^N$. Textures are described by smooth maps $\Sigma \to Gr_{\mathbb{C}}(M, N)$.
- Introduce physically relevant anisotropies. But then, we have to give up the holomorphic nature of the textures. Is there a well-behaved perturbation theory around this holomorphic $SU(N)$-symmetric limit ?
- Quantum melting of Skyrmion lattices far away from integer ν and transition to fractional quantum Hall states ?